

# The Coleman-Weinberg effective potential in the theory of superconductivity

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## Abstract

A quasi two-dimensional non-relativistic four-Fermi theory is studied at finite temperatures in the next-to-leading order approximation using the Coleman-Weinberg effective potential. The appearance of an imaginary part to the one-loop correction is discussed in the context of condensed matter theory where it is referred to as the Thouless criterion for superconductivity. By reference to the appropriate modified effective potential one may revise the Thouless criterion to obtain a critical temperature in next-to-leading order that, unlike the mean-field temperature, tends to zero in the two-dimensional limit in agreement with the Coleman theorem.

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# 1 Introduction

Low dimensional quantum field theories have recently attracted a great deal of attention. This is in part because the development of the theory of superconductivity applicable to high-temperature superconductors (HTSC) [1] demonstrated the need for theoretical methods which go beyond the standard Bardeen-Cooper-Schrieffer (BCS) [2] mean-field approximation. Within the BCS theory the fluctuations, or in other words next-to-leading order corrections are regarded as a small correction to the mean-field (leading-order) results. As such they do not lead to significant renormalization of such important characteristics as the critical temperature,  $T_c$ , of the superconducting transition.

By contrast, the influence of fluctuations in nonconventional superconductors may result in the significant decrease of  $T_c$ . The physical reason for this is rather simple. HTSC have, in contrast to conventional superconductors, a lower dimensionality of space for carrier motion and a smaller carrier density. It is known that these factors increase the influence of fluctuations.

Alternatively one can say that for low dimensional systems the leading-order approximation gives qualitatively wrong results. This happens, for example, when one studies the finite temperature 2+1 dimensional Nambu-Jona-Lasinio [3] or related (see [4, 5]) models which possess a continuous global symmetry. The leading-order calculation indicates that the symmetry remains broken for a range of non-zero temperatures. This result seems to contradict the predictions of the Coleman-Mermin-Wagner-Hohenberg theorem [6]. The argument is well known. The infrared region of the system is dominated by the zero Matsubara mode of the boson field. The boson field effectively resides in a 1+1 dimensional space. Thus, by appealing to the above mentioned theorem, one concludes that the symmetry breaking is forbidden. However, to show this explicitly one has to calculate the next-to-leading order correction. In the quasi-2D case [5] the situation is not as dramatic because the symmetry breaking transition is not forbidden, but nonetheless the next-to-leading order correction changes the results drastically.

Therefore one needs theoretical methods which allow one to study models beyond the mean-field approximation. This may be most simply achieved by adapting the methods developed in quantum field theory e.g. the Coleman-Weinberg method [7]. The main goal of the article is to show how the ap-

proach based on the Coleman-Weinberg effective potential may be used in the theory of superconductivity.

It is interesting to note that, as in the case of the well known phenomenon of dynamical symmetry breaking which was originally discovered in condensed matter theory, one observes that the Coleman-Weinberg method is in some sense predated by the Thouless approach to superconductivity [8].

There is, however, a crucial difference between the Coleman-Weinberg and Thouless methods. In the Thouless approach, the appearance of an imaginary part in the effective potential which occurs below the mean-field transition temperature,  $T_c^{MF}$ , is considered as the signature of the onset of superconductivity. This is the so called Thouless criterion of superconductivity which is, in fact, equivalent to the criterion given by the BCS theory. Contrary to this, a deeper analysis of the Coleman-Weinberg effective potential [9] (see also [10, 11]) has shown that the appearance of an imaginary part in the effective potential does not necessarily imply a symmetry breaking, e.g. superconducting transition. It only indicates a failure in the approximations used to derive the potential. It should prove useful and interesting to apply these concepts to the theory of superconductivity.

In Section 2 we present the model and derive the tree-potential along with the Coleman-Weinberg one-loop correction. The explanation of how the BCS mean-field results are related to the tree-potential allows one to understand easily why the one-loop correction is complex at the point of interest. The problem of complexity is known to appear in quantum field theories with spontaneous symmetry breaking at tree-level, and to correct it, one should replace the one loop correction by the so-called modified effective potential [9]. In Section 3 we approximate the solution to the full gap equation (including the fluctuations) for the case  $T_c \lesssim T_c^{MF}$  [12] and thus derive a corresponding modified effective potential. The solution of the full gap equation for the case  $T_c \ll T_c^{MF}$  is obtained in Section 4 and our conclusions are given in Section 5.

## 2 Model and Formalism

We study the following Hamiltonian density

$$H = -\psi_{\sigma}^{\dagger}(x) \left[ \frac{\nabla_{\perp}^2}{2m_{\perp}} + \frac{1}{m_z d^2} \cos(id\nabla_z) + \mu \right] \psi_{\sigma}(x) - V \psi_{\uparrow}^{\dagger}(x) \psi_{\downarrow}^{\dagger}(x) \psi_{\downarrow}(x) \psi_{\uparrow}(x), \quad (2.1)$$

which describes a layered quasi-2D superconductor with coherent interlayer tunneling. Here  $x \equiv \tau, \mathbf{r}_{\perp}, r_z$  (with imaginary time  $\tau$  and  $\mathbf{r}_{\perp}$  being a 2D vector);  $\psi_{\sigma}(x)$  is a fermion field,  $\sigma = \uparrow, \downarrow$  is the spin variable;  $m_{\perp}$  is the effective carrier mass in the planes (for example  $\text{CuO}_2$  planes);  $m_z$  is an effective mass in the  $z$ -direction;  $d$  is the interlayer distance;  $V$  is an effective local attraction constant;  $\mu$  is a chemical potential which fixes the carrier density  $n_f$ ; and we take  $\hbar = k_B = 1$ .

The relevance of the Hamiltonian (2.1) to the description of HTSC is discussed in [12, 13] and the references therein.

From a field theoretical point of view the closest model to the one considered here was studied in [5]. This model is the quasi-(2+1) dimensional Nambu-Jona-Lasinio model which has a very similar and rather rich phase diagram [5]. However, it is not our purpose here to discuss the whole phase diagram of our model here as was done in [12, 13] and we will restrict ourselves to the field theoretical aspects of the transition to the phase with broken symmetry. There is however an important difference between the model considered in [5] and the nonrelativistic model here. Due to the presence in (2.1) of the chemical potential  $\mu$  the density of particles in this model can be varied and may be large while in [5] the fermion density is not fixed. It is also known that in the presence of the Fermi surface the symmetry breaking transition may happen for arbitrarily small attraction, while in the Nambu-Jona-Lasinio model the attraction must be stronger than some critical value.

Note that only the case  $T_c \ll T_c^{MF}$  was considered in [5] (recall that  $T_c^{MF}$  is the mean-field transition temperature and  $T_c$  is the critical temperature of the superconducting transition). This is related to the assumption that the anisotropy of the quasi-2D system is sufficiently high. However, in the present work, one cannot make this assumption since, if the anisotropy of the system is not very large, one expects that  $T_c \lesssim T_c^{MF}$  for high carrier density [14, 15]. Thus we have to study here both the possibilities  $T_c \lesssim T_c^{MF}$  [12] and  $T_c \ll T_c^{MF}$ .

The standard Hubbard-Stratonovich method was used to study the Hamiltonian (2.1) (see for example the review [16]). In this method the statistical sum  $Z(v, \mu, T)$  ( $v$  is the volume of the system) is formally rewritten as a functional integral over the auxiliary Hubbard-Stratonovich fields  $\Phi$  and  $\Phi^*$

$$Z(v, \mu, T) = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \exp[-\beta\Omega(v, \mu, T, \Phi(x), \Phi^*(x))], \quad (2.2)$$

where

$$\beta\Omega(v, \mu, T, \Phi(x), \Phi^*(x)) = \frac{1}{V} \int_0^\beta d\tau \int d\mathbf{r} |\Phi(x)|^2 - \text{Tr} \text{Ln} G^{-1}[\Phi(x), \Phi^*(x)] \quad (2.3)$$

is the one-loop effective action. Here the auxiliary fields are given at the classical level by the equations of motion  $\Phi = V\psi_\downarrow\psi_\uparrow$  and  $\Phi^* = V\psi_\uparrow^\dagger\psi_\downarrow^\dagger$ .

The action (2.3) is expressed in terms of the Green function  $G$  which has in the Nambu representation (see e.g. [2]) the following operator form

$$G^{-1}[\Phi(x), \Phi^*(x)] = -\hat{I}\partial_\tau + \tau_3 \left[ \frac{\nabla_\perp^2}{2m_\perp} + \frac{1}{m_z d^2} \cos(id\nabla_z) + \mu \right] + \tau_+ \Phi(x) + \tau_- \Phi^*(x), \quad (2.4)$$

where  $\tau_3$ ,  $\tau_\pm = (\tau_1 \pm i\tau_2)/2$  are Pauli matrices. Although the representation (2.2), (2.3) is exact, in practical calculations it is necessary to restrict ourselves to some approximation. For our purposes the most convenient approximation is the Coleman-Weinberg [7] (see also [10, 11]) effective potential in the one-loop approximation. The exact expression (2.2) is replaced by

$$Z(v, \mu, T, |\Phi|^2) = \exp[-\beta\Omega_{pot}(v, \mu, T, |\Phi|^2)], \quad (2.5)$$

where the effective thermodynamical potential

$$\Omega_{pot}(v, \mu, T, |\Phi|^2) \simeq \Omega_{pot}^{\text{MF}}(v, \mu, T, |\Phi|^2) + \Omega^{(1)}(v, \mu, T, |\Phi|^2) \quad (2.6)$$

is expressed through the mean-field “tree-potential”

$$\Omega_{pot}^{\text{MF}}(v, \mu, T, |\Phi|^2) = \Omega(v, \mu, T, \Phi(x), \Phi^*(x))|_{\Phi, \Phi^* = \text{const}} \quad (2.7)$$

which may be evaluated explicitly to give

$$\Omega_{pot}^{\text{MF}}(v, \mu, T, |\Phi|^2) = v \left[ \frac{|\Phi|^2}{V} - \int \frac{d\mathbf{k}}{(2\pi)^3} \left\{ 2T \ln \cosh \frac{\sqrt{\xi^2(\mathbf{k}) + |\Phi|^2}}{2T} - \xi(\mathbf{k}) \right\} \right], \quad (2.8)$$

where

$$\xi(\mathbf{k}) = \mathbf{k}_\perp^2/2m_\perp - w \cos k_z d - \mu, \quad w = (m_z d^2)^{-1} \quad (2.9)$$

and through the one-loop (quantum) correction

$$\Omega^{(1)}(v, \mu, T, |\Phi|^2) = \frac{T}{2} \text{Tr} \text{Ln} \Gamma^{-1}(\tau, \mathbf{r})|_{\Phi, \Phi^* = \text{const}} \quad (2.10)$$

with

$$\Gamma^{-1}(\tau, \mathbf{r}) = \begin{pmatrix} \frac{\beta \delta^2 \Omega}{\delta \Phi^*(\tau, \mathbf{r}) \delta \Phi(0, \mathbf{0})} & \frac{\beta \delta^2 \Omega}{\delta \Phi(\tau, \mathbf{r}) \delta \Phi(0, \mathbf{0})} \\ \frac{\beta \delta^2 \Omega}{\delta \Phi^*(\tau, \mathbf{r}) \delta \Phi^*(0, \mathbf{0})} & \frac{\beta \delta^2 \Omega}{\delta \Phi(\tau, \mathbf{r}) \delta \Phi^*(0, \mathbf{0})} \end{pmatrix}. \quad (2.11)$$

Here and above the Tr and Ln are understood in the functional sense.

Using the “tree-potential” (2.8) one can easily reproduce the results of the BCS theory. Indeed the condition of minimization

$$\left. \frac{\partial \Omega_{\text{pot}}^{\text{MF}}(v, \mu, T, |\Phi|^2)}{\partial |\Phi|^2} \right|_{|\Phi| = \Phi_{\min}^{(0)}} = 0 \quad (2.12)$$

gives the standard BCS gap equation

$$\frac{1}{V} = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2\sqrt{\xi^2(\mathbf{k}) + (\Phi_{\min}^{(0)})^2}} \tanh \frac{\sqrt{\xi^2(\mathbf{k}) + (\Phi_{\min}^{(0)})^2}}{2T}, \quad (2.13)$$

where  $\xi(\mathbf{k})$  is given by (2.9).

In addition to the gap equation (2.12) there is the condition

$$-\frac{1}{v} \frac{\partial \Omega_{\text{pot}}^{\text{MF}}(v, \mu, T, |\Phi|^2)}{\partial \mu} = n_f, \quad (2.14)$$

which fixes the carrier density  $n_f$ . Since we have a quasi-2D system with a quadratic dispersion law in the planes the Fermi energy  $\epsilon_F$  is given by

$$\epsilon_F = \frac{\pi n_F d}{m_\perp}. \quad (2.15)$$

In the standard BCS theory the carrier density is so high that the feedback effect from the formation of the superconducting order parameter  $\Phi$  on the

chemical potential  $\mu$  is negligible and the equation (2.14) has a trivial solution  $\mu = \epsilon_F$ . Although for small carrier densities the equation (2.14) becomes very important (see for the review [17]), we consider here the high density limit only.

In this limit one can approximately split the quantum correction (2.10) written using the matrix (2.11) into the following sum

$$\Omega^{(1)}(v, \mu, T, |\Phi|^2) = \frac{T}{2} \text{Tr} \text{Ln} \Gamma_-^{-1}(\tau, \mathbf{r}) + \frac{T}{2} \text{Tr} \text{Ln} \Gamma_+^{-1}(\tau, \mathbf{r}), \quad (2.16)$$

where the Green functions  $\Gamma_{\pm}$  are

$$\begin{aligned} \Gamma_{\pm}^{-1}(\tau, \mathbf{r}) = & \frac{1}{V} \delta(\tau) \delta(\mathbf{r}) + \text{tr}[G(\tau, \mathbf{r}) \tau_+ G(-\tau, -\mathbf{r}) \tau_-] \\ & \pm \text{tr}[G(\tau, \mathbf{r}) \tau_- G(-\tau, -\mathbf{r}) \tau_-] |_{\Phi=\Phi^*=\text{const}}. \end{aligned} \quad (2.17)$$

This splitting is valid if these Green functions are even functions of momenta and frequency in the momentum representation. It was shown in [18] that this assumption is justified when one uses these functions in the derivative approximation, i.e. when they are expanded for small momenta and energy. This is the case considered in what follows.

In the weak coupling limit for local attraction between carriers it is appropriate to replace the attraction constant  $V$  by the two-particle bound state energy in vacuum [19, 20],

$$\varepsilon_b = -2W \exp\left(-\frac{4\pi d}{m_{\perp} V}\right). \quad (2.18)$$

Here  $W$  is the bandwidth in the plane and the limit  $V \rightarrow 0$ ,  $W \rightarrow \infty$  is to be understood. This replacement enables one to regularize the ultraviolet divergences which are present in the four-Fermi theory. Recall that in the case of non-local phonon attraction they are usually removed by the introduction of a cutoff at the Debye frequency [2]. This simplifies our condensed matter problem since one always has a natural scale for regularization.

Let us recall how the shape of the potential  $\Omega_{pot}^{MF}(|\Phi|^2)$  and the solution of Eq.(2.12) depend on  $T$ . Above the mean-field critical temperature  $T_c^{MF}$  the equation (2.12) only has the trivial solution  $\Phi = \Phi^* = 0$ , i.e. the “tree-potential” (2.7) is everywhere convex i.e.  $\partial^2 \Omega_{pot}^{MF} / \partial \Phi \partial \Phi^* > 0$ , with a minimum at  $\Phi = \Phi^* = 0$ .

The mean-field temperature,  $T_c^{MF}$ , is defined by the following equation

$$\left. \frac{\partial \Omega_{pot}^{MF}(v, \mu, T_c^{MF}, |\Phi|^2)}{\partial |\Phi|^2} \right|_{\Phi=\Phi^*=0} = 0, \quad (2.19)$$

which results in the standard BCS equation

$$\frac{1}{V} = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2\xi(\mathbf{k})} \tanh \frac{\xi(\mathbf{k})}{2T_c^{MF}}. \quad (2.20)$$

Taking into account the renormalization by (2.18) described above and using  $(m_z d^2)^{-1} \ll T_c^{MF}$  one obtains from (2.20) that

$$T_c^{MF} = \frac{\gamma}{\pi} \sqrt{2|\varepsilon_b|\epsilon_F}, \quad (2.21)$$

where  $\ln \gamma = 0.577$  is the Euler constant.

Thus at the temperature  $T_c^{MF}$  the curvature of the “tree-potential” around the point  $\Phi = \Phi^* = 0$  changes from convex to non-convex, i.e.

$$\partial^2 \Omega_{pot}^{MF}(T_c^{MF}) / \partial \Phi \partial \Phi^* |_{\Phi=\Phi^*=0} = 0.$$

Finally, for  $T < T_c^{MF}$  the equation (2.12) has a nontrivial degenerate solution which, for  $T \lesssim T_c^{MF}$ , satisfies

$$|\Phi(T)|^2 = \Phi_{min}^{(0)}(T)^2 = \frac{8\pi^2(T_c^{MF})^2}{7\zeta(3)} \left(1 - \frac{T}{T_c^{MF}}\right), \quad (2.22)$$

where  $\zeta(3)$  is the zeta function. This in turn means that  $\Omega_{pot}^{MF}(|\Phi|^2)$  has a degenerate minimum for which  $|\Phi(T)| = \Phi_{min}^{(0)}(T) \neq 0$ , while the point  $\Phi = \Phi^* = 0$  becomes a maximum so that the “tree-potential” is non-convex around this point,  $\partial^2 \Omega_{pot}^{MF} / \partial \Phi \partial \Phi^* |_{\Phi=\Phi^*=0} < 0$ .

A gap equation which already includes the fluctuation correction (2.16) has the same form as Eq.(2.12), but with the “tree-potential” replaced by the full potential (2.6):

$$\frac{\partial \Omega_{pot}(v, \mu, T, |\Phi|^2)}{\partial |\Phi|^2} = 0. \quad (2.23)$$

Correspondingly, the critical temperature,  $T_c$  is defined by

$$\left. \frac{\partial \Omega_{pot}(v, \mu, T_c, |\Phi|^2)}{\partial |\Phi|^2} \right|_{\Phi=\Phi^*=0} = 0. \quad (2.24)$$



Since  $T_c < T_c^{MF}$  it follows that the “tree-potential” is concave around the point  $\Phi = \Phi^* = 0$  at  $T_c$ . Nonetheless the full potential  $\Omega_{pot}$  must be convex i.e.  $\partial^2 \Omega_{pot} / \partial \Phi \partial \Phi^*|_{\Phi=\Phi^*=0} > 0$  around the point  $\Phi = \Phi^* = 0$  for  $T > T_c$  since otherwise  $T_c$  would not be the critical temperature. It follows from this that for  $T_c < T < T_c^{MF}$  the quantum correction (2.16) has to transform the maxima at the point  $\Phi = \Phi^* = 0$  to the minima of the full potential  $\Omega_{pot}(|\Phi|^2)$ .

Unfortunately, the one-loop correction  $\Omega^{(1)}$  given by (2.16) is ill-defined (complex) at the point of interest. The potential develops a complex part as the temperature drops below  $T = T_c^{MF}$  and in the earlier development of the BCS theory [8] this was considered as the sign of superconductivity (the Thouless criterion of superconductivity). However, this situation is rather standard in quantum field theory if one considers the class of theories with tree-level symmetry breaking and an extensive literature exists [9, 10, 21, 22]. Thus from a field theoretical point of view, the appearance of the imaginary part of the effective potential is only an indication that the one-loop approximation fails near the point  $\Phi = \Phi^* = 0$  and does not mean the symmetry is broken in the next-to-leading order approximation. Furthermore the imaginary part is related to the non-convexity of  $\Omega_{pot}^{MF}(|\Phi|^2)$  at this point which makes the argument of logarithm in (2.16) in the momentum space negative and this non-convexity is related with the symmetry breaking on the tree level. It is important to note that the discussion in [9, 10, 22] is valid only when the quantum correction is written in the diagonal form (2.16) rather than for the nondiagonal case (2.10).

There are many ways to circumvent the complexity of the effective potential explained above. For example, one can use the so-called Gaussian [21] or modified [9] effective potential which coincides with  $\Omega_{pot}$  in the region where the latter is well-defined and is still well-defined in the region where the original effective potential is ill-defined.

It is possible to find the modified potential directly, but we will use here, in our opinion, a more transparent consideration which will allow us to evaluate the one-loop correction to the gap equation not at  $\Phi = 0$  where it is ill-defined but in the region where it is well-defined. This will be done in the next section where the solution to the full gap equation for  $T_c \lesssim T_c^{MF}$  is obtained. We will also discuss there how our corresponding modified potential relates to that introduced in [9]. For simplicity we may assume from now on that  $\Phi$  is real without loss of generality.

### 3 The solution for $T_c \lesssim T_c^{MF}$

Let us assume that  $T_c \lesssim T_c^{MF}$  which means the position  $\Phi_{min}^{(0)}$  of the minimum of  $\Omega_{pot}^{MF}(|\Phi|^2)$  is close to zero. At this point  $\Omega_{pot}^{MF}$  is definitely convex and the one-loop correction (2.16) is real and well-defined. Thus for  $T_c \lesssim T_c^{MF}$  one can approximate Eq.(2.24) by

$$\frac{1}{v} \frac{\partial \Omega_{pot}^{MF}(v, \epsilon_F, T_c, \Phi^2)}{\partial \Phi^2} \Big|_{\Phi=0} + \frac{1}{v} \frac{\partial \Omega^{(1)}(v, \epsilon_F, T_c, \Phi^2)}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}} = 0, \quad (3.1)$$

where the value of  $\Phi_{min}^{(0)}$  at temperature  $T \lesssim T_c^{MF}$  is given by (2.22).

Therefore to solve the approximated gap equation (3.1) one has to calculate

$$\begin{aligned} & \frac{1}{v} \frac{\partial \Omega^{(1)}(v, \epsilon_F, T, \Phi^2)}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}} \\ &= \frac{1}{2} \frac{T}{(2\pi)^3} \sum_{\pm} \sum_{n=-\infty}^{\infty} \int d\mathbf{K} \Gamma_{\pm}(i\Omega_n, \mathbf{K}) \frac{\partial \Gamma_{\pm}^{-1}(i\Omega_n, \mathbf{K})}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}}, \end{aligned} \quad (3.2)$$

where  $\mathbf{K} = (\mathbf{K}_{\perp}, K_z)$ . Starting from (2.17), one can obtain the Green's functions as a function of  $\Phi$  in the momentum representation as

$$\Gamma_{\pm}^{-1}(i\Omega_n, \mathbf{K}) = \frac{1}{V} + \frac{T}{(2\pi)^3} \sum_{l=-\infty}^{\infty} \int d\mathbf{k} \frac{[i\omega_l - \xi_-][i\omega_l + i\Omega_n + \xi_+] \pm \Phi^2}{[\omega_l^2 + \xi_-^2 + \Phi^2][(\omega_l + \Omega_n)^2 + \xi_+^2 + \Phi^2]}, \quad (3.3)$$

where we have introduced the abbreviations  $\xi_{\pm} = \xi(\mathbf{k} \pm \mathbf{K}/2)$  and  $\Omega_n = 2\pi nT$ ,  $\omega_l = \pi(2l+1)T$  are odd and even Matsubara frequencies, respectively.

Since  $\Gamma_{-}^{-1}(0, \mathbf{0}) = 0$  is nothing but the BCS gap equation, the solution to the equation is the BCS value  $\Phi_{min}^{(0)}$ . In other words at  $\Phi = \Phi_{min}^{(0)}$ ,  $\Gamma_{-}^{-1}(0, \mathbf{K})$  has a zero at  $\mathbf{K} = 0$ . This gives rise to a pole in  $\Gamma_{-}(0, \mathbf{K})$  at  $\mathbf{K} = 0$  which is the only singular term in (3.2) for  $\Phi = \Phi_{min}^{(0)}$ .

$$\frac{1}{v} \frac{\partial \Omega^{(1)}(v, \epsilon_F, T, \Phi^2)}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}} \simeq \frac{1}{2} \frac{T}{(2\pi)^3} \int d\mathbf{K} \Gamma_{-}(0, \mathbf{K}) \Big|_{\Phi=\Phi_{min}^{(0)}} \frac{\partial \Gamma_{-}^{-1}(0, \mathbf{K})}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}} \quad (3.4)$$

In order to perform the calculations analytically we use the high-temperature derivative expansion for the Green functions  $\Gamma_{\pm}^{-1}$  [12],

$$\Gamma_{\pm}^{-1}(\Phi^2; 0, \mathbf{K}) = \frac{m_{\perp}}{2\pi d} \left[ \ln \left( \frac{T_c}{T_c^{MF}} \right) + a\mathbf{K}_{\perp}^2 + b(1 - \cos K_z d) + c(2\Phi^2 \pm \Phi^2) \right], \quad (3.5)$$

where

$$a = \frac{7\zeta(3)}{(4\pi)^2} \frac{\epsilon_F}{m_{\perp} T^2}, \quad b = \frac{7\zeta(3)}{(4\pi)^2} \frac{w^2}{T^2}, \quad c = \frac{7\zeta(3)}{8\pi^2 T^2}. \quad (3.6)$$

One can check that  $\Gamma_{-}^{-1}(0, \mathbf{0})$  becomes negative for  $\Phi < \Phi_{min}^{(0)}$  and this results in the appearance of the imaginary part of the effective potential described in the previous section.

Strictly speaking the one-loop correction (2.10) should be also real in the region

$\Phi_{conv} < \Phi < \Phi_{min}^{(0)}$ , where  $\Phi_{conv}$  is the point where the convexity of  $\Omega_{pot}^{MF}(|\Phi|^2)$  changes sign. However, due to the approximation implicit in the factorization needed to obtain (2.16) one observes that the one-loop correction is complex for  $\Phi < \Phi_{min}^{(0)}$ . This is the reason why we used in the approximated gap equation (3.1) the point  $\Phi_{min}^{(0)}$  for the calculation of the value of the effective potential rather than the point  $\Phi_{conv}$ .

Substituting (3.5) with  $\Phi_{min}^{(0)}$  given by (2.22) into (3.4), one arrives at the following approximation

$$\frac{1}{v} \frac{\partial \Omega^{(1)}(v, \epsilon_F, T, \Phi^2)}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}} \simeq \frac{T}{(2\pi)^3} \int d\mathbf{K} \frac{c}{a\mathbf{K}_{\perp}^2 + b[1 - \cos K_z d]}. \quad (3.7)$$

One can see that Eq.(3.7) has no infrared divergencies due to the presence of the third direction ( $b \neq 0$ ). In two dimensions it would be infrared divergent as required by the 2D theorems [6]. This equation also has an artificial ultraviolet divergence as a result of the replacement of the Green's function  $\Gamma$  by its derivative approximation. Thus one should introduce a rather natural ultraviolet cutoff  $(\mathbf{K}_{\perp}^{max})^2 = 2m_{\perp}\Phi(T=0) = 2m_{\perp}\sqrt{2|\varepsilon_b|\epsilon_F}$  and integrate over the momentum  $\mathbf{K}$  to obtain the expression

$$\frac{1}{v} \frac{\partial \Omega^{(1)}(v, \epsilon_F, T, \Phi^2)}{\partial \Phi^2} \Big|_{\Phi=\Phi_{min}^{(0)}} \simeq \frac{m_{\perp}}{2\pi d} \frac{T}{2\epsilon_F} |\ln \kappa|, \quad (3.8)$$

where

$$\kappa = \frac{1}{4\sqrt{2}} \frac{w^2}{\epsilon_F^2} \sqrt{\frac{\epsilon_F}{|\epsilon_b|}}. \quad (3.9)$$

Substituting (3.8) into (3.1) one obtains the final transcendental equation for  $T_c$

$$\ln \frac{T_c}{T_c^{MF}} + \frac{T_c}{2\epsilon_F} |\ln \kappa| = 0, \quad (3.10)$$

which may be rewritten in the following more convenient form

$$T_c = 2\epsilon_F \frac{|\ln(T_c/T_c^{MF})|}{|\ln \kappa|}. \quad (3.11)$$

One can see  $T_c$  goes to zero as  $m_z \rightarrow \infty$  ( $w \rightarrow 0$ ) as it must [6].

Strictly speaking the equation (3.11) is only valid when  $T_c \lesssim T_c^{MF}$ . Therefore one may expand the logarithm in equation (3.11) to obtain the equation

$$T_c = T_c^{MF} \left( 1 + \frac{T_c^{MF} |\ln \kappa|}{2\epsilon_F} \right)^{-1}. \quad (3.12)$$

As stated above one can also understand the approximation used in (3.1) in terms of the modified effective potential defined in [7]. The modified effective potential in [7] is defined as the minimum value for  $\Omega$  given a homogeneous state where  $|\Phi|^2$  is uniform. The real part of this modified potential has the following form

$$\tilde{\Omega}^{(1)}(v, \mu, T, |\Phi|^2) = \frac{T}{2(2\pi)^3} \sum_{\pm} \sum_{n=-\infty}^{\infty} \int_{\mathcal{D}} d\mathbf{K} \ln \Gamma_{\pm}^{-1}(i\Omega_n, \mathbf{K}), \quad (3.13)$$

where the area  $\mathcal{D}$  of integration in the momentum space includes only positive modes. One can see that (3.13) indeed coincides with (2.16) when  $\Omega^{(1)}$  is well-defined. Furthermore the modified potential (3.13) leads to the gap equation (3.1) which was considered above as the approximated one.

In the region  $\Phi < \Phi_{min}^{(0)}$  the modified effective potential considered above differs from the traditional effective potential,  $\Omega_{eff}(\Phi)$ , which is defined as the minimum value for  $\Omega$  such that the space average of  $\Phi(x)$  is given by  $\Phi$ . It can be shown that the conventional effective potential is in fact the convex envelope of the modified effective potential and is real and convex everywhere.

However for  $\Phi < \Phi_{min}^{(0)}$  it describes an inhomogeneous mixed state where the value of  $\Phi(x)$  is not uniform in space. One can readily understand that the modified and not the original potential is relevant for the superconducting state.

There is, however, the difference between our interpretation of the modified potential and that of [9]. In [9] the homogeneous state described by (3.13) is considered as decaying and the rate of the decay is related to negative modes of (2.16) which are not included in (3.13). It is physically obvious that there is no real decay of the homogeneous superconducting state with  $\Phi < \Phi_{min}^{(0)}$  for  $T < T_c$  although we have not been able to prove this rigorously. The absence of decay is in agreement with the interpretation of [21] although it should be stressed that the modified potential discussed here is not identical to the Gaussian effective potential in [21].

## 4 The solution for $T_c \ll T_c^{MF}$

In this section we find the approximate solution of Eq.(2.24) for the case  $T_c \ll T_c^{MF}$ , i.e. when the anisotropy is large ( $w/|\varepsilon_b| \ll 1$ ).

Following [5] we plan to expand about the minimum of the mean-field potential (2.7) i.e. about the point  $\Phi_{min}^{(0)}$ , which is determined by Eq.(2.12). Including the fluctuations shifts the minimum at  $\Phi_{min}^{(0)}$  to  $\Phi = \Phi_{min}^{(0)} + \Phi^{(1)}$ . One may therefore approximate (2.23), expanding about the mean-field minimum, as

$$\frac{\partial \Omega_{pot}(\Phi^2)}{\partial \Phi^2} \simeq \frac{\partial \Omega_{pot}(\sigma)}{\partial \sigma} + \frac{\partial^2 \Omega_{pot}(\sigma)}{(\partial \sigma)^2} 2\Phi_{min}^{(0)} \Phi^{(1)} = 0, \quad (4.1)$$

where we have introduced the short-hand notation  $\sigma \equiv (\Phi_{min}^{(0)})^2$ . This equation can be simplified using the “tree” gap equation (2.12) and one arrives at

$$\Phi^{(1)} = -\frac{\partial \Omega^{(1)}(\sigma)}{\partial \sigma} \left\{ 2\Phi_{min}^{(0)} \left[ \frac{\partial^2 \Omega_{pot}^{MF}(\sigma)}{(\partial \sigma)^2} + \frac{\partial^2 \Omega^{(1)}(\sigma)}{(\partial \sigma)^2} \right] \right\}^{-1}. \quad (4.2)$$

The temperature  $T_c$  is defined by the condition

$$\Phi(T_c) = \Phi_{min}^{(0)}(T_c) + \Phi^{(1)}(T_c) = 0. \quad (4.3)$$

Substituting  $\Phi_{min}^{(0)}(T_c)$  from (2.12) and  $\Phi^{(1)}$  from (4.2) into (4.3) one arrives at the equation for  $T_c$  in the same approximation as in [5]. It can be easily shown that the limiting behaviour  $T_c \rightarrow 0$  when  $m_z \rightarrow \infty$  is related to the behaviour of  $\partial\Omega^1(\sigma)/\partial\sigma$  (see below). For this reason for large  $m_z$  one can replace  $\Phi_{min}^{(0)}(T_c, m_z)$  and  $\Omega_{pot}^{MF}(T_c, m_z)$  by their 2D values at zero temperature,  $\Phi^{(0)}(T=0, m_z \rightarrow \infty)$  and  $\Omega_{pot}^{MF}(T=0, m_z \rightarrow \infty)$ , namely

$$\Phi_{min}^{(0)}(T_c, m_z) \simeq \sqrt{2|\varepsilon_b|\epsilon_F} \quad (4.4)$$

and

$$\frac{\partial^2 \Omega_{pot}^{MF}(\sigma)}{(\partial\sigma)^2} \simeq v \frac{m_\perp}{4\pi d} \frac{1}{|\varepsilon_b|(2\epsilon_F + |\varepsilon_b|)}, \quad (4.5)$$

where instead of  $V$  we used again the two-body bound state energy  $\varepsilon_b$  (2.18). One can check that the second derivative of  $\Omega^{(1)}$  is negligible relative to the corresponding derivative of  $\Omega_{pot}^{MF}$ . Thus one need only calculate the first derivative of  $\Omega^{(1)}$ .

The potential  $\Omega^{(1)}$  defined by (2.16) contains two terms: one involving the Green function  $\Gamma_+$  and one involving  $\Gamma_-$ . In the low temperature region and for  $\Phi_{min}^{(0)}$  defined by (2.22) the  $\Gamma_+$ -excitations are massive, while the  $\Gamma_-$ -excitations are massless (the Goldstone mode) [18]. Thus the gapped  $\Gamma_+$ -excitations are irrelevant and one can safely consider only the  $\Gamma_-$  part to obtain

$$\frac{\partial\Omega^{(1)}(\sigma)}{\partial\sigma} = \frac{1}{2} \frac{vT}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \int d\mathbf{K} \Gamma_-(i\Omega_n, \mathbf{K}) \frac{\partial\Gamma_-(i\Omega_n, \mathbf{K})}{\partial\sigma}, \quad (4.6)$$

where the Green function  $\Gamma_-(i\Omega_n, \mathbf{K})$  has been given in (3.3).

Since one knows that  $\Gamma_-(0, \mathbf{0}) = 0$  for  $\Phi = \Phi_{min}^{(0)}$ , as discussed after (3.3), the main contribution to the integral is concentrated near zero and it can be evaluated approximately as

$$\frac{\partial\Omega^{(1)}(\sigma)}{\partial\sigma} \simeq \frac{1}{2} \frac{vT}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \int d\mathbf{K} \Gamma_-(i\Omega_n, \mathbf{K}) \left. \frac{\partial\Gamma_-(0, \mathbf{0})}{\partial\sigma} \right|_{m_z \rightarrow \infty, T=0}, \quad (4.7)$$

where we have left  $m_z$  finite and  $T \neq 0$  only where necessary.

Differentiating (3.3) one arrives at

$$\begin{aligned} \left. \frac{\partial \Gamma_-(0, \mathbf{0})}{\partial \sigma} \right|_{m_z \rightarrow \infty, T=0} &= T \sum_{l=-\infty}^{\infty} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{[\omega_l^2 + \xi^2(\mathbf{k}) + \sigma]^2} \Big|_{m_z \rightarrow \infty, T=0} \\ &= \frac{1}{4\pi} \frac{m_{\perp}}{d} \frac{1}{(\Phi_{min}^{(0)})^2}. \end{aligned} \quad (4.8)$$

Thus one needs only the expression for  $\Gamma_-(i\Omega_n, \mathbf{K})$ . In the low temperature limit this Green function for small  $\Omega_n$  and  $\mathbf{K}$  (the derivative approximation) takes the following form

$$\Gamma_-^{-1}(i\Omega_n, \mathbf{K}) = a' \mathbf{K}_{\perp}^2 + b' K_z^2 + d' \Omega_n^2, \quad \Omega_n, \frac{\mathbf{K}_{\perp}^2}{m_{\perp}}, \frac{K_z^2}{m_z} \ll \Phi_{min}^{(0)}(T=0) \quad (4.9)$$

with

$$a' = \frac{\epsilon_F}{8\pi d (\Phi_{min}^{(0)})^2}, \quad b' = \frac{1}{32\pi d} \frac{m_{\perp}}{m_z} \frac{w}{\epsilon_F^2}, \quad d' = \frac{m_{\perp}}{8\pi d (\Phi_{min}^{(0)})^2}. \quad (4.10)$$

To obtain (4.9) one has to remove the ultraviolet divergences from (3.3) by applying the procedure of regularization via the two-body bound state energy described in Section 2. We note that in the limit  $T \rightarrow 0$  one can expand  $\Gamma_-^{-1}$  over  $\Omega_n$  without performing the analytical continuation to real frequency  $\Omega$ . Note also that after this analytic continuation is performed the pole of  $\Gamma_{R-}^{-1}(\Omega, \mathbf{K}_{\perp}, K_z = 0) = 0$  gives the correct dispersion relation for the Bogolyubov mode

$$\Omega = \sqrt{\epsilon_F/m_{\perp}} K_{\perp} = v_F/\sqrt{2} K_{\perp}. \quad (4.11)$$

Using the simple expression (4.9) makes it possible to calculate analytically  $\partial \Omega^{(1)}/\partial \sigma$  given by (4.7).

One can easily see that, in the 2D case when  $b' = 0$ , the term in the integral (4.7) with  $\Omega_n = 0$  is infrared divergent. This in turn implies that the gap equation (4.3) only has the trivial solution  $T_c \equiv 0$  which again demonstrates the absence of long range order in 2D [6].

If one uses the expansion (4.9), the equation (4.7) also has an artificial ultraviolet divergence. This is simply related to the use of the derivative expansion (4.9) which is only valid if  $\Omega_n$  and  $\mathbf{K}$  are small. One may use the same natural energy cutoff as in the previous case, namely the BCS gap  $\Phi_{min}^{(0)}(T=0)$ , to eliminate this divergence.

In the limit of large  $m_z$  the only relevant term in the sum (4.7) is  $\Omega_n = 0$  since this is the divergent term in the 2D limit. Using the abovementioned cutoff one arrives at

$$\frac{1}{2} \frac{vT}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \int d\mathbf{K} \Gamma_{-}(i\Omega_n, \mathbf{K}) \simeq vT \frac{(\Phi_{min}^{(0)})^2}{\epsilon_F} |\ln \delta|, \quad (4.12)$$

where

$$\delta = \frac{\pi^2}{4\sqrt{2}} \sqrt{\frac{|\epsilon_b|}{\epsilon_F}} \frac{w^2}{\epsilon_F^2}. \quad (4.13)$$

Substituting (4.12) into (4.7) and then (4.7) into (4.2) yields

$$\Phi^{(1)}(T) \simeq -\frac{T}{\epsilon_F} |\ln \delta| \frac{|\epsilon_b|(2\epsilon_F + |\epsilon_b|)}{2\Phi_{min}^{(0)}} \quad (4.14)$$

Finally substituting (4.14) into (4.3) one obtains the expression for the critical temperature as

$$T_c \simeq \frac{2\epsilon_F}{|\ln \delta|}. \quad (4.15)$$

This result is valid only for  $T_c \ll T_c^{MF}$  i.e. only when  $\delta \ll 1$ .

One can see that the expressions (3.11) and (4.15) both display logarithmic singularities in the limit  $m_z \rightarrow \infty$ . Also in both cases the leading order dependence on the Fermi energy  $\epsilon_F$  is linear. This is in contrast to the square root mean-field BCS dependence (2.21) and resembles the experimentally observed dependence [23]. Thus our two rather different approximations give qualitatively the same results.

## 5 Conclusion

To summarize, the appearance of an imaginary part in the one-loop effective potential does not signal the onset of superconductivity. Instead it reflects a well-known failure of the one-loop approximation. By reference to the modified effective potential one may derive a new approximation to the critical temperature in the limit  $T_c \lesssim T_c^{MF}$ . One may also derive an approximation to the critical temperature in the "zero-temperature" limit. Unlike the mean-field critical temperature both these approximate critical temperatures tend



to zero in the 2D limit in agreement with the Coleman-Mermin-Wagner-Hohenberg theorem [6]. In addition they both display a roughly linear dependence on the Fermi energy in agreement with experiment.

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